Algebraic Aspects of Propositional Logic

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Outline

This talk will focus on the connections between abstract algebra and propositional logic in a somewhat generalized setting. In particular:

- A brief description of Universal Algebra.
- How we can derive an algebra from a propositional theory of logic, where we want to make **elements in our algebra equal when they are logically equivalent in our theory**.

This talk is based on part of my MSc, which I completed this year in the UCT mathematics department.

A Boolean algebra is a set, B, which has a number of operations (eg. $\vee$), special elements (eg.1 or $\top$), such that these operations fulfill certain properties for every element of B. For example, for all $x, y \in B$

$$x \wedge y = y \wedge x$$

These properties can be expressed as equalities or “identities” between members of the Boolean algebra.

In fact, we can break down many algebraic structures (eg. vector spaces, groups, rings) into operations, special elements and identities. Universal algebra considers this in a general case.

Universal Algebra

We construct two special sets, Ω and Φ [1]:

- We define a set Ω, called a *signature*, which details the operations and special elements for our algebra. It also specifies how many inputs (eg. binary, unary) each operation acts on.
- Using Ω, we define the "terms" of our algebra, by taking a set of basic variables and joining them via operations. **Eg.** if x and y are variables, then $x \wedge y, \neg(x \vee y) \wedge y$ are terms.
- We then define the set Φ of *identities*, which consist of ordered pairs of terms. These tell us what terms should be equal in our algebra. **Eg.** if we want $x \wedge y = y \wedge x$ for all elements x, y, we include $(x \wedge y, y \wedge x)$ in our set of identities Φ.

Universal Algebras

A **general** (Ω, Φ)**-algebra** is a set A where we can define all the operations in Ω in such a way that all the identities in Φ are equal in A.[1]

Eg. if B is Boolean, we can define the "and" operation $\wedge$ such that $x \wedge y = y \wedge x$, or $x \wedge x = x$, for all $x, y \in B$.

When we fix a specific choice for Ω and Φ, the class of all (Ω, Φ)-algebras is called a **variety of universal algebra**.

It turns out that lots of mathematical structure can be analysed in a general variety.

We construct a (logical) **propositional theory**, $\mathscr{T}$, as follows:

1. We introduce set of symbols made up of *variables*, X and *logical connectives*, Ω.
2. We define the set of *formulae*, $\mathrm{Frm}(\mathscr{T})$, as follows:
   i. Any $x \in X$ is a formula.
   ii. Each logical connective allows us to create new formulae from old formulae, by connecting a fixed number of formulae together. Eg. if s, t are formulae, $s \wedge t$ and $\neg s$ are formulae.
3. We have a set of formulae which are considered *axioms*, and a number of specified *rules of inference*.

Deriving Algebraic Varieties from Propositional Theories

But hold on:

- The terms in universal algebra are constructed by variables acted upon by operations. The formulae in $\mathscr{T}$ are constructed by variables joined by connectives.
- Algebraic operations act on a certain number of elements. Logical connectives join a certain number of formulae.

And so we can construct an algebraic signature, Ω, out of the logical connectives in $\mathscr{T}$!

How do we make formulae equal when they are logically equivalent?

Using our axioms and applying our rules of inference, we are able to construct *proofs* in $\mathscr{T}$ which determine the *theorems* of our theory. We write $\vdash t$ to denote that t is a theorem of $\mathscr{T}$.[2]

If our theory has a connective signifying implication, $\Rightarrow$. Then we can consider formulae t and s logically equivalent when $\vdash t \Rightarrow s$ and $\vdash s \Rightarrow t$.

Deriving Varieties from Propositional Theories

This gives us the set of *identities* we require to make terms in our algebra equal! We simply let:

$$\Phi = \{(s, t) \mid\vdash s \Rightarrow t \text{ and } \vdash t \Rightarrow s\}$$

Thus, we are able to construct a signature, Ω, and a set of identities, Φ, where logically equivalent formulae are equal in our algebras. This determines a variety of (Ω, Φ)-algebras!

Examples [3]

1. Classical logic corresponds to the variety of Boolean algebras.
2. Intuitionistic logic corresponds to the variety of Heyting algebras. They have the same algebraic signature as Boolean algebras but have a weaker form of implication and negation which makes their structure a bit more tricky.
3. Normal modal logics correspond to a Boolean algebra with additional unary (1-ary)operations $\Box$ and $\Diamond$. Depending on the modal logic, these operations have properties resembling interior and closure operations in a field of mathematics called topology. In particular, we call the algebras corresponding to the modal logic **S4** topological Boolean algebras.

My Future Interests

I am still quite new to KR, but through my maths research I have become interested in pursuing further studies in non-classical logic in a more applied setting, such as:

- Non-monotonic reasoning and logics which incorporate uncertainty or belief change in general.
- The use of KR in explainable AI.
- Formal concept analysis and its applications.

References

1. Bergman, G. 2015. *An invitation to general algebra and universal constructions*. 2nd ed. Cham: Springer.
2. Mendelson, E. 1997. *Introduction to Mathematical Logic*. 4th ed. London: Chapman and Hall.
3. Rasiowa H. and Sikorski R. 1963. *The Mathematics of Metamathematics*. Warsawa: Polska Akademia Nauk, Monografie Matematyczne.

Thank you!